

Quantum symmetry groups of Hilbert modules equipped with orthogonal filtrations

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Abstract

We define and show the existence of the quantum symmetry group of a Hilbert module equipped with an orthogonal filtration. Our construction unifies the constructions of Banica-Skalski's quantum symmetry group of a C^* -algebra equipped with an orthogonal filtration and Goswami's quantum isometry group of an admissible spectral triple.

INTRODUCTION

The quantum isometry group of a noncommutative Riemannian compact manifold (an admissible spectral triple) was defined and constructed by Goswami in [6]. His breakthrough construction, technically more involved than the previous approaches to quantum symmetry groups in the case of finite structures [13, 3], provides a very natural direct link between Connes' noncommutative geometry [5] and the theory of compact quantum groups introduced by Woronowicz in the eighties [15]. We refer the reader to the introduction and bibliography of [4] for an overview of the several developments since Goswami's paper.

Motivated by the work of Goswami, Banica and Skalski define and construct in [4] the quantum symmetry group of a C^* -algebra endowed with an orthogonal filtration. Their construction provides a general powerful tool to define and check the existence of quantum symmetry groups of various mathematical systems and unifies several known quantum symmetry groups constructions. The work of Banica and Skalski also has the merit to clearly exhibit some of the structures needed to enable one to prove the existence of a compact quantum symmetry group, see [4] for details. However, although Goswami's work was one of the inspirations for [4], it seems that Goswami's quantum isometry group in [6] cannot, in general, be seen as a particular case of the quantum symmetry groups defined in [4] (because the subspace spanned by the eigenvalues of Goswami's Laplacian does not seem to form a subalgebra in general).

It is the purpose of the present paper to propose a construction that simultaneously generalizes the quantum symmetry groups of Goswami and of Banica-Skalski. We define and construct the quantum symmetry group of a Hilbert module endowed with an orthogonal filtration. The concept of Hilbert module endowed with an orthogonal filtration is inspired by Banica-Skalski's notion of C^* -algebra equipped with an orthogonal filtration, and is a natural generalization of it. Also, to an admissible spectral triple in the sense of [6], one can associate an appropriate Hilbert module endowed with an orthogonal filtration, and our quantum symmetry group coincides with the quantum isometry group in [6].

The concept also has the interest to provide an alternative approach to the quantum isometry group of a spectral triple. The main difference with the approach of [6] is that, instead of extracting from the spectral triple an analogue of the Laplacian on functions (the so-called “noncommutative Laplacian”) and making appropriate assumptions on its spectrum, we directly use the Dirac operator of the spectral triple, its spectrum and its natural domain. We then add assumptions to these data to get the desired orthogonally filtered Hilbert module. In the case of ordinary compact Riemannian manifolds, it is also possible to use our framework to provide an approach to the quantum isometry group, using the Hilbert module of continuous sections of the bundle of exterior forms, although we have not been able to show that our exterior forms based quantum isometry group coincides with the one of Goswami in [6] (it is always a quantum subgroup).

The paper is organized as follows. In the first part, we briefly recall some basic definitions about compact quantum groups. Then we introduce in part 2 the concept of Hilbert module endowed with an orthogonal filtration, and define the category of “quantum transformations groups” for a Hilbert module equipped with an orthogonal filtration (our starting point being the notion of action of a compact quantum group on a Hilbert module given in [1]). Part 3 is devoted to the proof of the existence of a universal object in this category. In the last part we discuss some examples and compare our construction with the ones of Goswami and Banica-Skalski mentioned previously.

Notations and conventions: By *algebra* we will always mean *unital algebra*. So that algebra morphisms are assumed to preserve the units. The symbol \odot will denote the algebraic tensor product, while \otimes will denote tensor product of maps, spatial tensor product of C^* -algebras, or exterior tensor product of Hilbert modules. If A is a C^* -algebra, we will denote by $\mathcal{S}(A)$ the set of states on A .

1. COMPACT QUANTUM GROUPS

We recall here some basic definitions on compact quantum groups. See [15, 16, 12] and [10] for more details.

Definition 1.1 – A *Woronowicz C^* -algebra* is a couple (Q, Δ) , where Q is a C^* -algebra and $\Delta : Q \rightarrow Q \otimes Q$ is a $*$ -morphism such that:

- $(\Delta \otimes id_Q) \circ \Delta = (id_Q \otimes \Delta) \circ \Delta,$

- the spaces $\text{span}\{\Delta(Q).(Q \otimes 1_Q)\}$ and $\text{span}\{\Delta(Q).(1_Q \otimes Q)\}$ are both dense in $Q \otimes Q$.

Definition 1.2 – Let (Q_0, Δ_0) and (Q_1, Δ_1) be Woronowicz C^* -algebras. A *morphism of Woronowicz C^* -algebras* from Q_0 to Q_1 is a $*$ -morphism:

$$\mu : Q_0 \rightarrow Q_1 \quad \text{such that} \quad (\mu \otimes \mu) \circ \Delta_0 = \Delta_1 \circ \mu.$$

The category of compact quantum groups is then defined to be the opposite category of the category of Woronowicz C^* -algebras.

Definitions 1.3 – Let $Q = (Q, \Delta)$ be a Woronowicz C^* -algebra.

- A *Woronowicz C^* -ideal* of Q is a C^* -ideal I of Q such that $\Delta(I) \subset \text{Ker}(\pi \otimes \pi)$, where $\pi : Q \rightarrow Q/I$ is the canonical quotient map.
- A *Woronowicz C^* -subalgebra* of Q is a C^* -subalgebra Q' of Q such that $\Delta(Q') \subset Q' \otimes Q'$.

Definition 1.4 – Let Q be a Woronowicz C^* -algebra. A matrix $(v_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(Q)$ is called *multiplicative* if we have $\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$ for all i, j .

The concept of an action of a quantum group on a C^* -algebra is formalized as follows.

Definition 1.5 – Let Q be a Woronowicz C^* -algebra and let A be a C^* -algebra. A *coaction of Q on A* is a $*$ -morphism $\alpha : A \rightarrow A \otimes Q$ satisfying:

- $(\alpha \otimes id_Q) \circ \alpha = (id_A \otimes \Delta) \circ \alpha$,
- $\text{span}\{\alpha(A).(1 \otimes Q)\}$ is dense in $A \otimes Q$.

We say that a coaction α of Q on A is *faithful* if there exists no nontrivial Woronowicz C^* -subalgebra Q' of Q such that $\alpha(A) \subset A \otimes Q'$. Furthermore if τ is a continuous linear functional on A , we say that α preserves τ if $(\tau \otimes id_Q) \circ \alpha = \tau(\cdot)1_Q$.

2. QUANTUM GROUPS ACTIONS ON HILBERT MODULES

We recall now the definition of an action of a compact quantum group on a Hilbert module (see [7] for background material on Hilbert modules). Then we introduce the notion of orthogonal filtration on a Hilbert module, give some natural examples of such objects, and define what we mean by *preserving the filtration* for an action of a compact quantum group on a Hilbert module endowed with an orthogonal filtration.

Definition 2.1 – Let A be a C^* -algebra. A *(right) pre-Hilbert A -module* is a vector space E , equipped with a (right) A -module structure together with an A -valued inner product $\langle \cdot | \cdot \rangle_A$, that is to say:

- $\forall \xi, \eta, \zeta \in E, \forall a, b \in A, \langle \xi | \eta a + \zeta b \rangle_A = \langle \xi | \eta \rangle_A a + \langle \xi | \zeta \rangle_A b$,

- $\forall \xi, \eta \in E, \langle \xi | \eta \rangle_A^* = \langle \eta | \xi \rangle_A,$
- $\forall \xi \in E, \langle \xi | \xi \rangle_A \geq 0$ and if $\langle \xi | \xi \rangle_A = 0$ then $\xi = 0$.

We define a norm $\|\cdot\|_A$ on E by setting for $\xi \in E, \|\xi\|_A = \|\langle \xi | \xi \rangle_A\|^{\frac{1}{2}}$. If furthermore E is complete with respect to this norm, we say that E is a (*right*) *Hilbert A -module*.

We say that E is *full* if the space $\langle E | E \rangle_A = \text{span}\{\langle \xi | \eta \rangle_A ; \xi, \eta \in E\}$ is dense in A .

Left Hilbert A -modules are defined analogously, except that the A -valued inner product ${}_A\langle \cdot | \cdot \rangle$ has to be linear in the first variable and antilinear in the second one. In what follows we will mostly consider right Hilbert modules. Of course, the construction can be adapted for left Hilbert modules.

The notion of coaction on a Hilbert module is due to Baaĵ and Skandalis [1, Definition 2.2]. But working with Woronowicz C^* -algebras instead of Hopf C^* -algebras simplifies the original definition:

Definition 2.2 – Let A be a C^* -algebra and let E be a Hilbert A -module. A *coaction* of a Woronowicz C^* -algebra Q on E consists of:

- a coaction $\alpha : A \rightarrow A \otimes Q,$
- a linear map $\beta : E \rightarrow E \otimes Q$ satisfying:
 - (a) $(\beta \otimes id_Q) \circ \beta = (id_E \otimes \Delta) \circ \beta,$
 - (b) $\text{span}\{\beta(E) \cdot (1 \otimes Q)\}$ is dense in $E \otimes Q,$
 - (c) $\forall \xi, \eta \in E, \langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes Q} = \alpha(\langle \xi | \eta \rangle_A),$
 - (d) $\forall \xi \in E, \forall a \in A, \beta(\xi \cdot a) = \beta(\xi) \cdot \alpha(a).$

We say that the coaction (α, β) of Q on E is faithful if there exists no nontrivial Woronowicz C^* -subalgebra Q' of Q such that $\beta(E) \subset E \otimes Q'$ (note that we do not require α to be faithful).

Remark 2.3 – If (α, β) is a coaction of a Woronowicz C^* -algebra Q on a Hilbert A -module E , then $\beta : E \rightarrow E \otimes Q$ is necessarily continuous. Indeed:

$$\text{For all } \xi \in E, \|\beta(\xi)\|_{A \otimes Q}^2 = \|\langle \beta(\xi) | \beta(\xi) \rangle_{A \otimes Q}\| = \|\alpha(\langle \xi | \xi \rangle_A)\| \leq \|\langle \xi | \xi \rangle_A\| = \|\xi\|_A^2.$$

Definition 2.4 – Let A be a C^* -algebra, let τ be a faithful state on A and let E be a Hilbert A -module. An *orthogonal filtration* $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ of E consists of:

- a family $(V_i)_{i \in \mathcal{I}}$ of finite-dimensional subspaces of E such that:
 - (a) for all $i, j \in \mathcal{I}$ with $i \neq j, \forall \xi \in V_i$ and $\forall \eta \in V_j, \tau(\langle \xi | \eta \rangle_A) = 0,$
 - (b) the space $\mathcal{E}_0 = \sum_{i \in \mathcal{I}} V_i$ is dense in $(E, \|\cdot\|_A),$
- an element $\xi_0 \in E,$
- a one-to-one antilinear operator $J : \mathcal{E}_0 \rightarrow E.$

Examples 2.5.

- (1) Let M be a compact Riemannian manifold. The space of continuous sections of the bundle of exterior forms on M , $\Gamma(\Lambda^*M)$, is a Hilbert $C(M)$ -module. We can equip it with an orthogonal filtration by taking $\tau = \int \cdot d\text{vol}$ (where $d\text{vol}$ denotes the Riemannian density of M), $\xi_0 = m \mapsto 1_{\Lambda_m^*M}$, $J : \Gamma(\Lambda^*M) \rightarrow \Gamma(\Lambda^*M)$ the canonical involution and $(V_i)_{i \in \mathbb{N}}$ the family of eigenspaces of the de Rham operator $D = \overline{d} + d^*$.
- (2) We recall from [4] the definition of a C^* -algebra equipped with an orthogonal filtration:

Definition 2.6 – Let A be a C^* -algebra, τ be a faithful state on A and $(V_i)_{i \in \mathcal{I}}$ be a family of finite-dimensional subspaces of A (with the index set \mathcal{I} containing a distinguished element 0). We say that $(\tau, (V_i)_{i \in \mathcal{I}})$ is an *orthogonal filtration* of A if:

- (a) $V_0 = \mathbb{C} \cdot 1_A$,
- (b) $\forall i, j \in \mathcal{I}$ such that $i \neq j$, $\forall a \in V_i$ and $\forall b \in V_j$, $\tau(a^*b) = 0$,
- (c) the space $\mathcal{A}_0 = \sum_{i \in \mathcal{I}} V_i$ is a dense $*$ -subalgebra of A .

Setting $E = A$ (with its canonical Hilbert A -module structure), $\xi_0 = 1_A$ and $J = a \mapsto a^*$, then $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ is an orthogonal filtration of E .

- (3) Let $(\mathcal{A}, \mathcal{H}, D)$ be an admissible spectral triple in the sense of [6]. We set:

- (a) $E = A = \overline{\mathcal{A}}^{\mathcal{L}(\mathcal{H})}$,
- (b) $\tau = \begin{cases} a \mapsto \frac{\text{Tr}_\omega(a|D|^{-p})}{\text{Tr}_\omega(|D|^{-p})} & \text{if } \mathcal{H} \text{ is infinite dimensional,} \\ \text{the usual trace} & \text{otherwise,} \end{cases}$
where Tr_ω denotes the Dixmier trace and p is the metric dimension of $(\mathcal{A}, \mathcal{H}, D)$,
- (c) the $(V_i)_{i \in \mathbb{N}}$ are the eigenspaces of the ‘noncommutative Laplacian’,
- (d) ξ_0 and J are respectively the unit and the involution of A .

The couple $(\tau, (V_i)_{i \in \mathbb{N}})$ is not in general an orthogonal filtration of A in the sense of [4] since $\sum_{i \in \mathbb{N}} V_i$ is not necessarily a $*$ -subalgebra of A . However, $(\tau, (V_i)_{i \in \mathbb{N}}, J, \xi_0)$ is an orthogonal filtration of A , seen as a Hilbert A -module.

- (4) Let us recall some common conditions on spectral triples.

Definition 2.7 – Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with finite metric dimension p .

- We say that $(\mathcal{A}, \mathcal{H}, D)$ *satisfies the finiteness and absolute continuity condition* if the space $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{N}} \text{Dom}(D^k)$ is a finitely generated projective left \mathcal{A} -module, and if there exists $q \in \mathcal{M}_n(\mathcal{A})$ with $q = q^2 = q^*$ such that:

- (a) $\mathcal{H}^\infty \cong \mathcal{A}^n q$,

- (b) the left \mathcal{A} -scalar product ${}_{\mathcal{A}}\langle \cdot | \cdot \rangle$ induced on \mathcal{H}^∞ by the previous isomorphism satisfies:

$$\frac{Tr_\omega({}_{\mathcal{A}}\langle \xi | \eta \rangle |D|^{-p})}{Tr_\omega(|D|^{-p})} = (\eta | \xi)_{\mathcal{H}}.$$

(Note that if $(\mathcal{A}, \mathcal{H}, D)$ is regular then \mathcal{H}^∞ is automatically a left \mathcal{A} -module.)

- We say that $(\mathcal{A}, \mathcal{H}, D)$ is *real* if it is equipped with an antiunitary operator $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$ such that $\mathcal{J}(\text{Dom}(D)) \subset \text{Dom}(D)$ and $\forall a, b \in \mathcal{A}, [a, \mathcal{J}b^* \mathcal{J}^*] = 0$.

If $(\mathcal{A}, \mathcal{H}, D)$ satisfies the finiteness and absolute continuity condition it is natural to consider $A = \overline{\mathcal{A}}^{\mathcal{L}(\mathcal{H})}$ and the Hilbert A -module E obtain by completing \mathcal{H}^∞ (for the A -norm). The eigenspaces $(V_i)_{i \in \mathbb{N}}$ of D are two by two orthogonal in \mathcal{H} , thus $\forall \xi \in V_i, \forall \eta \in V_j$ such that $i \neq j$, we get $\tau({}_{\mathcal{A}}\langle \xi | \eta \rangle) = 0$ where $\tau = a \mapsto \frac{Tr_\omega(a|D|^{-p})}{Tr_\omega(|D|^{-p})}$. If τ is faithful and \mathcal{E}_0 is dense in E , then E can be equipped with an orthogonal filtration (with $J : \mathcal{E}_0 \rightarrow \mathcal{E}_0$ any one-to-one antilinear map and e.g. $\xi_0 = 0$).

If we assume furthermore that $(\mathcal{A}, \mathcal{H}, D)$ is real, then a natural choice is to set $J = \mathcal{J}|_{\mathcal{E}_0}$.

Notation 2.8 – Let A be a C^* -algebra and let E be a Hilbert A -module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. We define on E a scalar product by:

$$\forall \xi, \eta \in E, (\xi | \eta)_\tau = \tau(\langle \xi | \eta \rangle_A).$$

We denote by \mathcal{H} the completion of E with respect to this scalar product and by $\|\cdot\|_\tau$ the norm associated with it.

Remark that $\mathcal{E}_0 = \bigoplus_{i \in \mathcal{I}} V_i \subset \mathcal{H}$ and since $\forall \xi \in E, \|\xi\|_\tau^2 = \tau(\langle \xi | \xi \rangle_A) \leq \|\langle \xi | \xi \rangle_A\| = \|\xi\|_A^2$ we have a continuous injection $E \hookrightarrow \mathcal{H}$ with dense image.

We will define now the coactions that preserve the structure of a given Hilbert module equipped with an orthogonal filtration. This will allow us to describe the category of its “quantum transformations groups”.

Definition 2.9 – Let E be a Hilbert A -module equipped with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. A *filtration preserving coaction* of a Woronowicz C^* -algebra Q on E is a coaction (α, β) of Q on E satisfying:

- $(\tau \otimes id_Q) \circ \alpha = \tau(\cdot)1_Q$,
- $\forall i \in \mathcal{I}, \beta(V_i) \subset V_i \odot Q$,
- $(J \otimes *) \circ \beta = \beta \circ J$ on \mathcal{E}_0 , where $*$ denotes the involution of A ,
- $\beta(\xi_0) = \xi_0 \otimes 1_Q$.

In this case, we will also say that Q coacts on E in a *filtration preserving way*.

Definition 2.10 – Let E be a Hilbert A -module equipped with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. We will denote by $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ the category of Woronowicz C^* -algebras coacting on E in a filtration preserving way. If (α_0, β_0) and (α_1, β_1) are filtration preserving coactions of Woronowicz C^* -algebras Q_0 and Q_1 on E , then a morphism from Q_0 to Q_1 in this category is a morphism of Woronowicz C^* -algebras $\mu : Q_0 \rightarrow Q_1$ satisfying:

$$\alpha_1 = (id_A \otimes \mu) \circ \alpha_0 \quad \text{and} \quad \beta_1 = (id_E \otimes \mu) \circ \beta_0.$$

Remark 2.11 – If E is full and $\mu : Q_0 \rightarrow Q_1$ is a morphism of Woronowicz C^* -algebras satisfying $\beta_1 = (id_E \otimes \mu) \circ \beta_0$, then μ automatically satisfies $\alpha_1 = (id_A \otimes \mu) \circ \alpha_0$.

Indeed, for all $\xi, \eta \in E$:

$$\begin{aligned} \alpha_1(\langle \xi | \eta \rangle_A) &= \langle \beta_1(\xi) | \beta_1(\eta) \rangle_{A \otimes Q_1} = \langle (id_E \otimes \mu) \circ \beta_0(\xi) | (id_E \otimes \mu) \circ \beta_0(\eta) \rangle_{A \otimes Q_1} \\ &= (id_A \otimes \mu)(\langle \beta_0(\xi) | \beta_0(\eta) \rangle_{A \otimes Q_0}) = (id_A \otimes \mu) \circ \alpha_0(\langle \xi | \eta \rangle_A). \end{aligned}$$

And since E is full, we get $\alpha_1 = (id_A \otimes \mu) \circ \alpha_0$.

Remark 2.12 – When $E = \Gamma(\Lambda^* M)$ is equipped with the orthogonal filtration $(\tau, (V_i)_{i \in \mathbb{N}}, J, \xi_0)$ described in example 2.5.(1), the full subcategory of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ consisting of commutative Woronowicz C^* -algebras coacting on $\Gamma(\Lambda^* M)$ in a filtration preserving way is antiequivalent to the category of compact groups acting isometrically on M (see section 4.2 for more details). This explains our choice of seeing the opposite category of $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ as the category of quantum transformations groups of E . Moreover since the isometry group of M is a universal object in the category of compact groups acting isometrically on M , we will define the quantum symmetry group of E as a universal object in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. Proving the existence of such a universal object is the aim of the next section.

3. CONSTRUCTION OF THE QUANTUM SYMMETRY GROUP OF A HILBERT MODULE EQUIPPED WITH AN ORTHOGONAL FILTRATION

The following theorem generalizes the results of Goswami [6] and Banica-Skalski [4].

Theorem 3.1 – *Let A be a C^* -algebra and let E be a full Hilbert A -module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. The category $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ admits an initial object, which means that there exists a universal Woronowicz C^* -algebra coacting on E in a filtration preserving way. The quantum group corresponding to that universal object will be called the **quantum symmetry group** of $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$.*

Examples will be discussed in the next section. This section is devoted to the proof of Theorem 3.1. The proof mostly consists in carefully adapting Goswami's arguments in [6, Section 4]. In what follows E denotes a full Hilbert module over a given C^* -algebra A , equipped with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$.

Lemma 3.2 – Let (α, β) be a filtration preserving coaction of a Woronowicz C^* -algebra Q on E . The Q -linear map $\bar{\beta} : \mathcal{E}_0 \odot Q \rightarrow \mathcal{E}_0 \odot Q$ given by $\bar{\beta}(\xi \otimes x) = \beta(\xi)(1 \otimes x)$ extends to a unitary of the Hilbert Q -module $\mathcal{H} \otimes Q$.

PROOF. We have for $\xi, \eta \in \mathcal{E}_0$ and $x, y \in Q$:

$$\begin{aligned}
\langle \bar{\beta}(\xi \otimes x) | \bar{\beta}(\eta \otimes y) \rangle_Q &= x^* \langle \beta(\xi) | \beta(\eta) \rangle_Q y \\
&= x^* (\tau \otimes id) (\langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes Q}) y \quad (\text{by definition of } (\cdot | \cdot)_\tau = \tau(\langle \cdot | \cdot \rangle_A)) \\
&= x^* (\tau \otimes id) \circ \alpha (\langle \xi | \eta \rangle_A) y \\
&= x^* \tau (\langle \xi | \eta \rangle_A) y \quad (\text{since } (\tau \otimes id) \circ \alpha = \tau(\cdot) 1_Q) \\
&= (\xi | \eta)_\tau x^* y = \langle \xi \otimes x | \eta \otimes y \rangle_Q.
\end{aligned}$$

In particular $\bar{\beta}$ is isometric and thus extends to a Q -linear isometric operator still denoted by $\bar{\beta} : \mathcal{H} \otimes Q \rightarrow \mathcal{H} \otimes Q$. To show that $\bar{\beta}$ is unitary, it is enough to check that $\bar{\beta}$ has dense image. Since $\text{span}\{\beta(E).(1 \otimes Q)\}$ is dense in $E \otimes Q$ and \mathcal{E}_0 is dense in E , it follows that $\text{span}\{\beta(\mathcal{E}_0).(1 \otimes Q)\}$ is dense in $E \otimes Q$. Moreover the canonical injection $E \otimes Q \hookrightarrow \mathcal{H} \otimes Q$ has dense image, so that $\text{span}\{\beta(\mathcal{E}_0).(1 \otimes Q)\}$ is also dense in $\mathcal{H} \otimes Q$. \square

Notation 3.3 – We define on \mathcal{E}_0 a left scalar product by:

$$\tau(\xi | \eta) = \tau(\langle J(\xi) | J(\eta) \rangle_A).$$

For each $i \in \mathcal{I}$ we set $d_i = \dim(V_i)$ and we fix:

- an orthonormal basis $(e_{ij})_{1 \leq j \leq d_i}$ of V_i for the right scalar product $(\cdot | \cdot)_\tau$,
- an orthonormal basis $(f_{ij})_{1 \leq j \leq d_i}$ of V_i for the left scalar product $\tau(\cdot | \cdot)$.

We denote by $p^{(i)} \in GL_{d_i}(\mathbb{C})$ the change of basis matrix from (f_{ij}) to the basis (e_{ij}) of V_i and we set $s^{(i)} = p^{(i)t} \overline{p^{(i)}}$.

Lemma 3.4 – Let (α, β) be a filtration preserving coaction of a Woronowicz C^* -algebra Q on E . For all $i \in \mathcal{I}$, we denote by $v^{(i)}$ the multiplicative matrix associated with the basis $(e_{ij})_{1 \leq j \leq d_i}$ of the Q -comodule V_i (in other words, $v^{(i)}$ is characterized by: $\forall j, \beta(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{kj}^{(i)}$).

- For all $i \in \mathcal{I}$, the matrix $v^{(i)} = (v_{kj}^{(i)})_{1 \leq k, j \leq d_i}$ is unitary and

$$v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} v^{(i)t} = I_{d_i}$$

- The unital C^* -subalgebra Q' of Q generated by $\{v_{kj}^{(i)} ; i \in \mathcal{I}, j, k \in \{1, \dots, d_i\}\}$ is a Woronowicz C^* -subalgebra of Q satisfying $\alpha(A) \subset A \otimes Q'$ and $\beta(E) \subset E \otimes Q'$. Furthermore (α, β) is a faithful filtration preserving coaction of Q' on E .

PROOF. First let us check that the $v^{(i)}$'s are unitary matrices.

Consider the unitary $\overline{\beta} : \mathcal{H} \otimes Q \rightarrow \mathcal{H} \otimes Q$ of the Hilbert Q -module $\mathcal{H} \otimes Q$ constructed in the previous lemma. For all i, j, m, n , we have $\langle e_{ij} \otimes 1 | \overline{\beta}(e_{mn} \otimes 1) \rangle_Q = \delta_{im} v_{jn}^{(i)} = \sum_{k=1}^{d_i} \langle e_{ik} \otimes v_{jk}^{(i)*} | e_{mn} \otimes 1 \rangle_Q$.

Thus for all i, j , we have $\overline{\beta}^*(e_{ij} \otimes 1) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{jk}^{(i)*}$. Then we get:

$$e_{ij} \otimes 1 = \overline{\beta} \circ \overline{\beta}^*(e_{ij} \otimes 1) = \sum_{k=1}^{d_i} \beta(e_{ik})(1 \otimes v_{jk}^{(i)*}) = \sum_{k,l=1}^{d_i} e_{il} \otimes v_{lk}^{(i)} v_{jk}^{(i)*},$$

which shows that for all $l, j \in \{1, \dots, d_i\}$, $\sum_{k=1}^{d_i} v_{lk}^{(i)} v_{jk}^{(i)*} = \delta_{lj}$, i.e. $v^{(i)} v^{(i)*} = I_{d_i}$.

Similarly (using $e_{ij} \otimes 1 = \overline{\beta}^* \circ \overline{\beta}(e_{ij} \otimes 1)$) we get $v^{(i)*} v^{(i)} = I_{d_i}$. Thus the matrices $v^{(i)}$ are unitary.

★ Let us show now that $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} v^{(i)t} = I_{d_i}$.

Let $i \in \mathcal{I}$. As $v^{(i)}$ is a multiplicative and unitary matrix in a Woronowicz C^* -algebra, the matrix $v^{(i)t}$ is invertible in $\mathcal{M}_{d_i}(Q)$ (cf. [9]), so it is enough to prove that $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = I_{d_i}$.

Using Sweedler's notations, we get for $\xi, \eta \in \mathcal{E}_0$:

$$\begin{aligned} {}_Q \langle \beta(\xi) | \beta(\eta) \rangle &= \sum \tau \left(\langle J(\xi_{(0)}) | J(\eta_{(0)}) \rangle_A \right) \xi_{(1)} \eta_{(1)}^* = \sum (\tau \otimes id) \left(\langle J(\xi_{(0)}) \otimes \xi_{(1)}^* | J(\eta_{(0)}) \otimes \eta_{(1)}^* \rangle_{A \otimes Q} \right) \\ &= (\tau \otimes id) \left(\langle (J \otimes *) \circ \beta(\xi) | (J \otimes *) \circ \beta(\eta) \rangle_{A \otimes Q} \right) = (\tau \otimes id) \left(\langle \beta \circ J(\xi) | \beta \circ J(\eta) \rangle_{A \otimes Q} \right) \\ &= (\tau \otimes id) \circ \alpha \left(\langle J(\xi) | J(\eta) \rangle_A \right) = \tau \left(\langle J(\xi) | J(\eta) \rangle_A \right) 1_Q = \tau(\xi | \eta) 1_Q. \end{aligned}$$

Moreover, since $\beta(V_i) \subset V_i \odot Q$, there exists $w^{(i)} \in \mathcal{M}_{d_i}(Q)$ such that $\forall j, \beta(f_{ij}) = \sum_{k=1}^{d_i} f_{ik} \otimes w_{kj}^{(i)}$.

Then we get ${}_Q \langle \beta(f_{ij}) | \beta(f_{ik}) \rangle = \sum_{l,m=1}^{d_i} \tau(f_{il} | f_{im}) w_{lj}^{(i)} w_{mk}^{(i)*} = \sum_{l=1}^{d_i} w_{lj}^{(i)} w_{lk}^{(i)*} = \tau(f_{ij} | f_{ik}) 1_Q = \delta_{jk}$.

This shows:

$$w^{(i)t} \overline{w^{(i)}} = I_{d_i}. \quad (1)$$

Furthermore we have for all j , $e_{ij} = \sum_{k=1}^{d_i} p_{kj}^{(i)} f_{ik}$, thus $w^{(i)} = p^{(i)} v^{(i)} (p^{(i)})^{-1}$. Then replacing $w^{(i)}$ in the equality (1) we get $(p^{(i)t})^{-1} v^{(i)t} p^{(i)t} \overline{p^{(i)} v^{(i)} (p^{(i)})^{-1}} = I_{d_i}$, which shows that $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = I_{d_i}$, where $s^{(i)} = p^{(i)t} \overline{p^{(i)}}$.

★ It then follows easily that Q' is a Woronowicz C^* -subalgebra of Q satisfying $\alpha(A) \subset A \otimes Q'$ and $\beta(E) \subset E \otimes Q'$. Indeed, since the $v^{(i)}$'s are multiplicative matrices, we have $\Delta(Q') \subset Q' \otimes Q'$ so that Q' is a Woronowicz C^* -subalgebra of Q . Moreover $\beta(\mathcal{E}_0) \subset \mathcal{E}_0 \odot Q'$, thus $\beta(E) \subset E \otimes Q'$. Then for all $\xi, \eta \in \mathcal{E}_0$, $\alpha(\langle \xi | \eta \rangle_A) = \langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes Q} \subset \langle E \otimes Q' | E \otimes Q' \rangle_{A \otimes Q} \subset A \otimes Q'$. This shows $\alpha(A) \subset A \otimes Q'$ since $\langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$ is dense in A .

★ It remains to check that (α, β) is a faithful filtration preserving coaction of Q' on E .

We only show that $\text{span}\{\alpha(A).(1 \otimes Q')\}$ and $\text{span}\{\beta(E).(1 \otimes Q')\}$ are respectively dense in $A \otimes Q'$ and $E \otimes Q'$ (the other conditions that must satisfy (α, β) to be a filtration preserving coaction of Q' on E directly follow from the fact that it is a filtration preserving coaction of Q on E).

We have for all i, j :

$$\begin{aligned}
\sum_{k=1}^{d_i} \beta(e_{ik})(1 \otimes v_{jk}^{(i)*}) &= \sum_{k,l=1}^{d_i} (e_{il} \otimes v_{lk}^{(i)})(1 \otimes v_{jk}^{(i)*}) = \sum_{k,l=1}^{d_i} e_{il} \otimes (v_{lk}^{(i)} v_{jk}^{(i)*}) \\
&= \sum_{l=1}^{d_i} e_{il} \otimes \delta_{lj} \quad (\text{since } v^{(i)} v^{(i)*} = I_{d_i}) \\
&= e_{ij} \otimes 1.
\end{aligned}$$

This implies that $\text{span}\{\beta(E).(1 \otimes Q')\}$ is dense in $E \otimes Q'$. Moreover, we have seen that for all $i \in \mathcal{I}$, $\overline{v^{(i)}}$ is invertible in $\mathcal{M}_{d_i}(Q')$ with inverse $x^{(i)} = (s^{(i)})^{-1} v^{(i)t} s^{(i)}$.

Let $i, j \in \mathcal{I}$, $m \in \{1, \dots, d_i\}$ and $n \in \{1, \dots, d_j\}$:

$$\begin{aligned}
\sum_{k=1}^{d_i} \sum_{l=1}^{d_j} \alpha(\langle e_{ik} | e_{jl} \rangle_A) \cdot (1 \otimes v_{nl}^{(j)*} x_{km}^{(i)}) &= \sum_{k=1}^{d_i} \sum_{l=1}^{d_j} \langle \beta(e_{ik}) | \beta(e_{jl}) \rangle_{A \otimes Q} \cdot (1 \otimes v_{nl}^{(j)*} x_{km}^{(i)}) \\
&= \sum_{k=1}^{d_i} \langle \beta(e_{ik}) | \underbrace{\sum_{l=1}^{d_j} \beta(e_{jl}) \cdot (1 \otimes v_{nl}^{(j)*})}_{e_{jn} \otimes 1} \rangle_{A \otimes Q} \cdot (1 \otimes x_{km}^{(i)}) \\
&= \sum_{k,l=1}^{d_i} \langle e_{il} \otimes v_{lk}^{(i)} | e_{jn} \otimes 1 \rangle_{A \otimes Q} \cdot (1 \otimes x_{km}^{(i)}) \\
&= \sum_{k,l=1}^{d_i} \langle e_{il} | e_{jn} \rangle_A \otimes (v_{lk}^{(i)*} x_{km}^{(i)}) \\
&= \sum_{l=1}^{d_i} \langle e_{il} | e_{jn} \rangle_A \otimes \delta_{lm} \quad (\text{car } \overline{v^{(i)}} x^{(i)} = I_{d_i}) \\
&= \langle e_{im} | e_{jn} \rangle_A \otimes 1.
\end{aligned}$$

Thus for all i, j, m, n , $\langle e_{im} | e_{jn} \rangle_A \otimes 1$ is in $\text{span}\{\alpha(A).(1 \otimes Q')\}$. By density of $\langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$ in A , this shows that $\text{span}\{\alpha(A).(1 \otimes Q')\}$ is dense in $A \otimes Q'$. \square

Notation 3.5 – For all $i \in \mathcal{I}$, we consider $A_u(s^{(i)})$ the universal Woronowicz C^* -algebra of Van Daele and Wang (see [11]) associated with $s^{(i)}$. That is, $A_u(s^{(i)})$ is the universal Woronowicz C^* -algebra generated by a multiplicative and unitary matrix $u^{(i)} = (u_{kj}^{(i)})_{1 \leq k, j \leq d_i}$, satisfying the following relations:

$$u^{(i)t} s^{(i)} \overline{u^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{u^{(i)}} (s^{(i)})^{-1} u^{(i)t} = I_{d_i}.$$

We set $\mathcal{U} = \bigstar_{i \in \mathcal{I}} A_u(s^{(i)})$ and $\beta_u : \mathcal{E}_0 \rightarrow \mathcal{E}_0 \odot \mathcal{U}$ the linear map given by:

$$\beta_u(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes u_{kj}^{(i)}.$$

See [12] for the construction of free product of compact quantum groups.

In the following, if (Q, Δ) is a Woronowicz C^* -algebra and I is a Woronowicz C^* -ideal of Q , we will denote by $\pi_I : Q \rightarrow Q/I$ the canonical projection and by Δ_I the canonical coproduct of Q/I (i.e. $\Delta_I : Q/I \rightarrow Q/I \otimes Q/I$ is the unique $*$ -morphism satisfying $\Delta_I \circ \pi_I = (\pi_I \otimes \pi_I) \circ \Delta$).

Lemma 3.6 – *Let (α, β) be a faithful filtration preserving coaction of a Woronowicz C^* -algebra Q on E . There exists a Woronowicz C^* -ideal $I \subset \mathcal{U}$ and a faithful filtration preserving coaction (α_I, β_I) of \mathcal{U}/I on E such that:*

- \mathcal{U}/I and Q are isomorphic in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$,
- β_I extends $(id \otimes \pi_I) \circ \beta_u$.

PROOF. For all $i \in \mathcal{I}$, we denote by $v^{(i)}$ the multiplicative matrix associated with the basis $(e_{ij})_{1 \leq j \leq d_i}$ of the Q -comodule V_i . In virtue of lemma 3.4, we know that $v^{(i)}$ is unitary and satisfies $v^{(i)t} s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{v^{(i)}} (s^{(i)})^{-1} v^{(i)t} = I_{d_i}$. So by universal property of \mathcal{U} there exists a morphism of Woronowicz C^* -algebras $\mu : \mathcal{U} \rightarrow Q$ such that for all i, p, q , $\mu(u_{pq}^{(i)}) = v_{pq}^{(i)}$. Then $\text{Im } \mu$ is a Woronowicz C^* -subalgebra of Q and for all i, j :

$$\beta(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{kj}^{(i)} = (id \otimes \mu) \circ \beta_u(e_{ij}). \quad (2)$$

Thus the inclusion $\beta(E) \subset E \otimes (\text{Im } \mu)$ holds, so that μ is necessarily onto (since the coaction is faithful). We set $I = \text{Ker } \mu$, we denote by $\hat{\mu} : \mathcal{U}/I \rightarrow Q$ the isomorphism of Woronowicz C^* -algebras such that $\hat{\mu} \circ \pi_I = \mu$ and we set $\alpha_I = (id \otimes \hat{\mu}^{-1}) \circ \alpha$ and $\beta_I = (id \otimes \hat{\mu}^{-1}) \circ \beta$. It is then easy to see that (α_I, β_I) is a filtration preserving coaction, and that:

$$\hat{\mu} : (\mathcal{U}/I, \Delta_I, \alpha_I, \beta_I) \rightarrow (Q, \Delta, \alpha, \beta) \text{ is an isomorphism.}$$

Thanks to (2), we see that $\beta_I = (id \otimes \hat{\mu}^{-1}) \circ \beta$ extends $(id \otimes \pi_I) \circ \beta_u$. □

Before proving Theorem 3.1 we need a last lemma.

Lemma 3.7 – *Let A and B be C^* -algebras and let \mathcal{J} be a nonempty family of C^* -ideals of B . Set $I_0 = \bigcap_{I \in \mathcal{J}} I$, and for $I \in \mathcal{J}$, set $p_I : B/I_0 \rightarrow B/I$ the unique $*$ -morphism such that:*

$$\begin{array}{ccc} & B & \\ \pi_0 \swarrow & & \searrow \pi_I \\ B/I_0 & \xrightarrow{p_I} & B/I \end{array}$$

where π_0 and π_I denote the canonical projections. Then we have, for all $x \in A \otimes (B/I_0)$:

$$\|x\| = \sup_{I \in \mathcal{J}} \|id \otimes p_I(x)\|.$$

PROOF. For $b \in B/I_0$ and $x \in A \otimes B/I_0$ we set:

$$\|b\|_\infty = \sup_{I \in \mathcal{I}} \|p_I(b)\| \quad \text{and} \quad \|x\|_\infty = \sup_{I \in \mathcal{I}} \|id \otimes p_I(x)\|.$$

This defines C^* -seminorms on B/I_0 and $A \otimes B/I_0$ respectively. In fact since $I_0 = \bigcap_{I \in \mathcal{I}} I$, we get that $\|\cdot\|_\infty$ is a C^* -norm on B/I_0 . So by uniqueness of the C^* -norm on B/I_0 , $\|\cdot\|_\infty$ coincides with the usual norm on B/I_0 . To prove the lemma, we will use the same argument on $A \otimes B/I_0$, thus we only have to show that $\|\cdot\|_\infty$ is a C^* -norm on $A \otimes B/I_0$.

Let x be in $A \otimes B/I_0$ such that $\|x\|_\infty = 0$. Then for all $I \in \mathcal{I}$, $(id \otimes p_I)(x) = 0$. Thus $(f \otimes (g \circ p_I))(x) = 0$ for all $f \in \mathcal{S}(A)$ and all $g \in \mathcal{S}(B/I)$. But the convex hull of the set $\mathcal{S}_0 = \{g \circ p_I ; I \in \mathcal{I}, g \in \mathcal{S}(B/I)\}$ is weak*-dense in $\mathcal{S}(B/I_0)$. Indeed, we have for every self-adjoint element $b \in B/I_0$:

$$\|b\| = \|b\|_\infty = \sup_{I \in \mathcal{I}} \|p_I(b)\| = \sup_{I \in \mathcal{I}, g \in \mathcal{S}(B/I)} |g \circ p_I(b)| = \sup_{h \in \mathcal{S}_0} |h(b)|$$

and it follows from [14, Lemma T.5.9] that $\mathcal{S}(B/I_0)$ is contained in the weak*-closed convex hull of \mathcal{S}_0 . Consequently, we have for all $f \in \mathcal{S}(A)$ and all $g \in \mathcal{S}(B/I_0)$, $(f \otimes g)(x) = 0$. Since any continuous linear functional on a C^* -algebra is a linear combination of states, we get $(f \otimes g)(x) = 0$ for all $f \in A^*$ and all $g \in (B/I_0)^*$. Then by [8, Proposition 3.2.11], we conclude that $x = 0$. \square

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. We denote by \mathcal{I} the set of all C^* -ideals $I \subset \mathcal{U}$ such that:

$(id \otimes \pi_I) \circ \beta_u$ extends to a continuous linear map $\beta_I : E \rightarrow E \otimes \mathcal{U}/I$ such that there exists a $*$ -morphism $\alpha_I : A \rightarrow A \otimes \mathcal{U}/I$ preserving τ and satisfying:

- $\forall \xi, \eta \in E, \langle \beta_I(\xi) | \beta_I(\eta) \rangle_{A \otimes \mathcal{U}/I} = \alpha_I(\langle \xi | \eta \rangle_A)$,
- $\forall \xi \in E, \forall a \in A, \beta_I(\xi \cdot a) = \beta_I(\xi) \cdot \alpha_I(a)$,
- $(J \otimes *) \circ \beta_I = \beta_I \circ J$ on \mathcal{E}_0 ,
- $\beta_I(\xi_0) = \xi_0 \otimes 1$.

The set \mathcal{I} is nonempty, since it contains the kernel of the counit $\varepsilon : \mathcal{U} \rightarrow \mathbb{C}$ (this can be directly checked, or seen by applying lemma 3.6 to the trivial coaction $A \rightarrow A \otimes \mathbb{C}, E \rightarrow E \otimes \mathbb{C}$).

We denote by I_0 the intersection of all elements of \mathcal{I} , by $Q_0 = \mathcal{U}/I_0$ and by $\pi_0 : \mathcal{U} \rightarrow Q_0$ the canonical projection (as intersection of C^* -ideals, I_0 is a C^* -ideal, so π_0 is a $*$ -morphism). Let us show that $I_0 \in \mathcal{I}$.

★ First let us show that $(id \otimes \pi_0) \circ \beta_u$ extends to a continuous linear map $\beta_0 : E \rightarrow E \otimes Q_0$. Note that for $x \in E \otimes Q_0$, $\|x\|_{A \otimes Q_0} = \sup_{I \in \mathcal{I}} \|id_E \otimes p_I(x)\|_{A \otimes \mathcal{U}/I}$. Indeed:

$$\begin{aligned} \|x\|_{A \otimes Q_0}^2 &= \|\langle x | x \rangle_{A \otimes Q_0}\| = \sup_{I \in \mathcal{I}} \|(id_A \otimes p_I)(\langle x | x \rangle_{A \otimes Q_0})\| \quad (\text{by the previous lemma}) \\ &= \sup_{I \in \mathcal{I}} \|\langle id_E \otimes p_I(x) | id_E \otimes p_I(x) \rangle_{A \otimes \mathcal{U}/I}\| = \sup_{I \in \mathcal{I}} \|id_E \otimes p_I(x)\|_{A \otimes \mathcal{U}/I}^2. \end{aligned}$$

Furthermore we have for all $\xi \in \mathcal{E}_0$ and all $I \in \mathcal{I}$:

$$\|(id_E \otimes \pi_I) \circ \beta_u(\xi)\|_{A \otimes \mathcal{U}/I}^2 = \|\langle \beta_I(\xi) | \beta_I(\xi) \rangle_{A \otimes \mathcal{U}/I}\| = \|\alpha_I(\langle \xi | \xi \rangle_A)\| \leq \|\langle \xi | \xi \rangle_A\| = \|\xi\|_A^2$$

since α_I is a $*$ -morphism. Hence for all $\xi \in \mathcal{E}_0$:

$$\begin{aligned} \|(id_E \otimes \pi_0) \circ \beta_u(\xi)\|_{A \otimes Q_0} &= \sup_{I \in \mathcal{I}} \|(id_E \otimes (\pi_I \circ \pi_0)) \circ \beta_u(\xi)\|_{A \otimes \mathcal{U}/I} \\ &= \sup_{I \in \mathcal{I}} \|(id_E \otimes \pi_I) \circ \beta_u(\xi)\|_{A \otimes \mathcal{U}/I} \leq \|\xi\|_A, \end{aligned}$$

which shows that $(id \otimes \pi_0) \circ \beta_u$ extends to a continuous linear map $\beta_0 : E \rightarrow E \otimes Q_0$.

★ Next let us show that there exists a linear map $\alpha_0 : \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A \rightarrow A \otimes Q_0$ such that

$$\forall \xi, \eta \in \mathcal{E}_0, \alpha_0(\langle \xi | \eta \rangle_A) = (id \otimes \pi_0) \left(\langle \beta_u(\xi) | \beta_u(\eta) \rangle_{A \otimes \mathcal{U}} \right).$$

Let ξ_1, \dots, ξ_n and η_1, \dots, η_n be elements of \mathcal{E}_0 such that $\sum_{i=1}^n \langle \xi_i | \eta_i \rangle_A = 0$.

Then for all $I \in \mathcal{I}$:

$$\sum_{i=1}^n (id \otimes \pi_I) \left(\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}} \right) = \sum_{i=1}^n \langle \beta_I(\xi_i) | \beta_I(\eta_i) \rangle_{A \otimes \mathcal{U}/I} = \sum_{i=1}^n \alpha_I(\langle \xi_i | \eta_i \rangle_A) = \alpha_I(0) = 0.$$

Thus we have:

$$\left\| \sum_{i=1}^n (id \otimes \pi_0) \left(\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}} \right) \right\| = \sup_{I \in \mathcal{I}} \left\| \sum_{i=1}^n (id \otimes \pi_I) \left(\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}} \right) \right\| = 0.$$

This shows that we can define a linear map $\alpha_0 : \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A \rightarrow A \otimes Q_0$ by the formula

$$\alpha_0 \left(\sum_{i=1}^n \langle \xi_i | \eta_i \rangle_A \right) = \sum_{i=1}^n (id \otimes \pi_0) \left(\langle \beta_u(\xi_i) | \beta_u(\eta_i) \rangle_{A \otimes \mathcal{U}} \right).$$

★ Let us check that $\alpha_0 : \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A \rightarrow A \otimes Q_0$ extends to a $*$ -morphism $\alpha_0 : A \rightarrow A \otimes Q_0$ preserving τ . We get for all $\xi, \eta \in \mathcal{E}_0$ and all I in \mathcal{I} :

$$\begin{aligned} (id \otimes p_I) \circ \alpha_0(\langle \xi | \eta \rangle_A) &= (id \otimes p_I) \circ (id \otimes \pi_0) \left(\langle \beta_u(\xi) | \beta_u(\eta) \rangle_{A \otimes \mathcal{U}} \right) \\ &= (id \otimes \pi_I) \left(\langle \beta_u(\xi) | \beta_u(\eta) \rangle_{A \otimes \mathcal{U}} \right) = \alpha_I(\langle \xi | \eta \rangle_A). \end{aligned}$$

Hence $(id \otimes p_I) \circ \alpha_0$ and α_I coincide on $\langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$. Consequently we have for $x \in \langle \mathcal{E}_0 | \mathcal{E}_0 \rangle_A$:

$$\|\alpha_0(x)\| = \sup_{I \in \mathcal{I}} \|(id \otimes p_I) \circ \alpha_0(x)\| = \sup_{I \in \mathcal{I}} \|\alpha_I(x)\| \leq \|x\|.$$

Thus α_0 extends continuously to A . Moreover for all $a, b \in A$, we have

$$\begin{aligned} (id \otimes p_I)(\alpha_0(ab) - \alpha_0(a)\alpha_0(b)) &= \alpha_I(ab) - \alpha_I(a)\alpha_I(b) = 0 \\ \text{and } (id \otimes p_I)(\alpha_0(a^*) - \alpha_0(a)^*) &= \alpha_I(a^*) - \alpha_I(a)^* = 0. \end{aligned}$$

Hence $\|\alpha_0(ab) - \alpha_0(a)\alpha_0(b)\| = \sup_{I \in \mathcal{I}} \|(id \otimes p_I)(\alpha_0(ab) - \alpha_0(a)\alpha_0(b))\| = 0$.

Similarly, we get $\|\alpha_0(a^*) - \alpha_0(a)^*\| = 0$. So α_0 is indeed a $*$ -morphism. Moreover for all $a \in A$ and all $I \in \mathcal{I}$, we have

$$\begin{aligned} p_I \circ (\tau \otimes id)(\alpha_0(a)) &= (\tau \otimes p_I)(\alpha_0(a)) = (\tau \otimes id) \circ \alpha_I(a) \\ &= \tau(a)1_{\mathcal{U}/I} = p_I(\tau(a)1_{Q_0}). \end{aligned}$$

Thus α_0 preserves τ .

★ We are now ready to check that $I_0 \in \mathcal{I}$. For all $\xi, \eta \in \mathcal{E}_0$ we have (by construction of α_0) that $\alpha_0(\langle \xi | \eta \rangle_A) = \langle \beta_0(\xi) | \beta_0(\eta) \rangle_{A \otimes Q_0}$, and this equality extends by continuity for $\xi, \eta \in E$. Since $(id_A \otimes p_I) \circ \alpha_0 = \alpha_I$ and $(id_E \otimes p_I) \circ \beta_0 = \beta_I$ for all $I \in \mathcal{I}$, we get for all $\xi \in E$ and $a \in A$:

$$\begin{aligned} \|\beta_0(\xi.a) - \beta_0(\xi).\alpha_0(a)\| &= \sup_{I \in \mathcal{I}} \|(id_E \otimes p_I) \circ \beta_0(\xi.a) - (id_E \otimes p_I)(\beta_0(\xi).\alpha_0(a))\| \\ &= \sup_{I \in \mathcal{I}} \|(id_E \otimes p_I) \circ \beta_0(\xi.a) - [(id_E \otimes p_I) \circ \beta_0(\xi)].[(id_A \otimes p_I) \circ \alpha_0(a)]\| \\ &= \sup_{I \in \mathcal{I}} \|\beta_I(\xi.a) - \beta_I(\xi).\alpha_I(a)\| = 0. \end{aligned}$$

Thus $\beta_0(\xi.a) = \beta_0(\xi).\alpha_0(a)$.

Similarly for $\xi \in \mathcal{E}_0$, $\|(J \otimes *) \circ \beta_0(\xi) - \beta_0 \circ J(\xi)\| = \sup_{I \in \mathcal{I}} \|(J \otimes *) \circ \beta_I(\xi) - \beta_I \circ J(\xi)\| = 0$ and $\|\beta_0(\xi_0) - \xi_0 \otimes 1_{Q_0}\| = \sup_{I \in \mathcal{I}} \|\beta_I(\xi_0) - \xi_0 \otimes 1_{\mathcal{U}/I}\| = 0$. Thus we have $(J \otimes *) \circ \beta_0 = \beta_0 \circ J$ on \mathcal{E}_0 and $\beta_0(\xi_0) = \xi_0 \otimes 1$. We conclude that $I_0 \in \mathcal{I}$.

We set $K = \text{Ker}((\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}})$. In order to show that I_0 is a Woronowicz C^* -ideal we have to check that $I_0 \subset K$, and by definition of I_0 it is enough to show that $K \in \mathcal{I}$.

Denote by $\mu : \mathcal{U}/K \rightarrow \text{Im}((\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}})$ the C^* -isomorphism that satisfies $(\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}} = \mu \circ \pi_K$. Then for all i, j :

$$\begin{aligned} (id \otimes \mu) \circ (id \otimes \pi_K) \circ \beta_u(e_{ij}) &= (id \otimes \pi_0 \otimes \pi_0) \circ (id \otimes \Delta_{\mathcal{U}}) \circ \beta_u(e_{ij}) \\ &= \sum_{k,l=1}^{d_i} e_{il} \otimes \pi_0(u_{lk}^{(i)}) \otimes \pi_0(u_{kj}^{(i)}) = \sum_{k=1}^{d_i} (\beta_0 \otimes id)(e_{ik} \otimes \pi_0(u_{kj}^{(i)})) \\ &= (\beta_0 \otimes id) \circ \beta_0(e_{ij}). \end{aligned} \tag{3}$$

Thus we have $(\beta_0 \otimes id) \circ \beta_0(E) \subset E \otimes \text{Im } \mu$, and we set $\beta_K = (id \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0 : E \rightarrow E \otimes \mathcal{U}/K$.

We get for all i, j, m, n :

$$\begin{aligned} (\alpha_0 \otimes id) \circ \alpha_0(\langle e_{ij} | e_{mn} \rangle_A) &= \alpha_0 \otimes id \left(\left\langle \sum_{k=1}^{d_i} e_{ik} \otimes \pi_0(u_{kj}^{(i)}) \middle| \sum_{l=1}^{d_m} e_{ml} \otimes \pi_0(u_{ln}^{(m)}) \right\rangle_{A \otimes Q_0} \right) \\ &= \alpha_0 \otimes id \left(\sum_{k,l} \langle e_{ik} | e_{ml} \rangle_A \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \right) \\ &= \sum_{k,l} \langle \beta_0(e_{ik}) | \beta_0(e_{ml}) \rangle_{A \otimes Q_0} \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \\ &= \sum_{k,l} \left\langle \sum_{p=1}^{d_i} e_{ip} \otimes \pi_0(u_{pk}^{(i)}) \middle| \sum_{q=1}^{d_m} e_{mq} \otimes \pi_0(u_{ql}^{(m)}) \right\rangle_{A \otimes Q_0} \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k,l,p,q} \langle e_{ip} | e_{mq} \rangle_A \otimes \pi_0(u_{pk}^{(i)*} u_{ql}^{(m)}) \otimes \pi_0(u_{kj}^{(i)*} u_{ln}^{(m)}) \\
&= \langle (\beta_0 \otimes id) \circ \beta_0(e_{ij}) | (\beta_0 \otimes id) \circ \beta_0(e_{ml}) \rangle_{A \otimes Q_0 \otimes Q_0} \\
&= (id \otimes \mu)(\langle \beta_K(e_{ij}) | \beta_K(e_{ml}) \rangle). \tag{4}
\end{aligned}$$

Hence for $\xi, \eta \in E$, we have

$$(\alpha_0 \otimes id) \circ \alpha_0(\langle \xi | \eta \rangle_A) = \langle (\beta_0 \otimes id) \circ \beta_0(\xi) | (\beta_0 \otimes id) \circ \beta_0(\eta) \rangle_{A \otimes Q_0 \otimes Q_0} \in A \otimes \text{Im } \mu.$$

Thus we also have $(\alpha_0 \otimes id) \circ \alpha_0(A) \subset A \otimes \text{Im } \mu$, and we define:

$$\alpha_K = (id \otimes \mu^{-1}) \circ (\alpha_0 \otimes id) \circ \alpha_0 : A \rightarrow A \otimes \mathcal{U}/K.$$

We know from (3) that β_K extends $(id \otimes \pi_K) \circ \beta_u$ and from (4) that for all $\xi, \eta \in E$, $\alpha_K(\langle \xi | \eta \rangle_A) = \langle \beta_K(\xi) | \beta_K(\eta) \rangle_{A \otimes \mathcal{U}/K}$.

★ Let us check that α_K preserves τ . We have for all $a \in A$:

$$\begin{aligned}
(\tau \otimes id) \circ \alpha_K(x) &= (\tau \otimes id_{\mathcal{U}/K}) \circ (id_A \otimes \mu^{-1}) \circ (\alpha_0 \otimes id_{Q_0}) \circ \alpha_0(x) \\
&= \mu^{-1} \circ (\tau \otimes id_{Q_0} \otimes id_{Q_0}) \circ (\alpha_0 \otimes id_{Q_0}) \circ \alpha_0(x) \\
&= \mu^{-1} \circ (\tau(\cdot)1_{Q_0} \otimes id_{Q_0}) \circ \alpha_0(x) = \mu^{-1}(1_{Q_0} \otimes (\tau \otimes id_{Q_0}) \circ \alpha_0(x)) \\
&= \mu^{-1}(1_{Q_0} \otimes \tau(x)1_{Q_0}) = \tau(x)1_{\mathcal{U}/K}.
\end{aligned}$$

★ We have for $\xi \in E$ and $a \in A$:

$$\begin{aligned}
\beta_K(\xi.a) &= (id \otimes \mu^{-1}) \circ (\beta_0 \otimes id)(\beta_0(\xi.a)) = (id \otimes \mu^{-1}) \circ (\beta_0 \otimes id)(\beta_0(\xi). \alpha_0(a)) \\
&= (id \otimes \mu^{-1})((\beta_0 \otimes id) \circ \beta_0(\xi).(\alpha_0 \otimes id) \circ \alpha_0(a)) = \beta_K(\xi). \alpha_K(a).
\end{aligned}$$

★ Moreover, we have on \mathcal{E}_0 :

$$\begin{aligned}
(J \otimes *) \circ \beta_K &= (J \otimes *) \circ (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0 = (id_E \otimes \mu^{-1}) \circ (J \otimes * \otimes *) \circ (\beta_0 \otimes id) \circ \beta_0 \\
&= (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ (J \otimes *) \circ \beta_0 = (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0 \circ J = \beta_K \circ J.
\end{aligned}$$

and $\beta_K(\xi_0) = (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id) \circ \beta_0(\xi_0) = (id_E \otimes \mu^{-1}) \circ (\beta_0 \otimes id)(\xi_0 \otimes 1)$

$$= (id_E \otimes \mu^{-1})(\xi_0 \otimes 1 \otimes 1) = \xi_0 \otimes 1.$$

So $K \in \mathcal{I}$ and I_0 is indeed a Woronowicz C^* -ideal. We denote by Δ_0 the coproduct on Q_0 . In order to show that $Q_0 \in \mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ it only remains to check that α_0 and β_0 are coassociative and that $\text{span}\{\alpha_0(A).(1 \otimes Q_0)\}$ and $\text{span}\{\beta_0(E).(1 \otimes Q_0)\}$ are respectively dense in $A \otimes Q_0$ and $E \otimes Q_0$.

★ We have seen (cf. (3)) that for all i, j , $(\beta_0 \otimes id) \circ \beta_0(e_{ij}) = (id \otimes \pi_0 \otimes \pi_0) \circ (id \otimes \Delta_{\mathcal{U}}) \circ \beta_u(e_{ij})$. But $(\pi_0 \otimes \pi_0) \circ \Delta_{\mathcal{U}} = \Delta_0 \circ \pi_0$. Thus:

$$(\beta_0 \otimes id) \circ \beta_0(e_{ij}) = (id \otimes \Delta_0) \circ (id \otimes \pi_0) \circ \beta_u(e_{ij}) = (id \otimes \Delta_0) \circ \beta_0(e_{ij}).$$

We deduce that $(\beta_0 \otimes id) \circ \beta_0 = (id \otimes \Delta_0) \circ \beta_0$ on E .

Hence for $\xi, \eta \in E$:

$$\begin{aligned} (\alpha_0 \otimes id_{Q_0}) \circ \alpha_0(\langle \xi | \eta \rangle_A) &= \langle (\beta_0 \otimes id_{Q_0}) \circ \beta_0(\xi) | (\beta_0 \otimes id_{Q_0}) \circ \beta_0(\eta) \rangle_{A \otimes Q_0 \otimes Q_0} \\ &= \langle (id_E \otimes \Delta_0) \circ \beta_0(\xi) | (id_E \otimes \Delta_0) \circ \beta_0(\eta) \rangle_{A \otimes Q_0 \otimes Q_0} \\ &= (id_A \otimes \Delta_0) \left(\langle \beta_0(\xi) | \beta_0(\eta) \rangle_{A \otimes Q_0} \right) \\ &= (id_A \otimes \Delta_0) \circ \alpha_0(\langle \xi | \eta \rangle_A), \end{aligned}$$

which shows (by density of $\langle E | E \rangle_A$ in A) that α_0 is coassociative as well.

★ Finally, to show that $\text{span}\{\alpha_0(A).(1 \otimes Q_0)\}$ and $\text{span}\{\beta_0(E).(1 \otimes Q_0)\}$ are respectively dense in $A \otimes Q_0$ and $E \otimes Q_0$, we can proceed in the same way as in the proof of lemma 3.4, by checking that for all i, j :

$$\sum_{k=1}^{d_i} \beta_0(e_{ik})(1 \otimes \pi_0(u_{jk}^{(i)*})) = e_{ij} \otimes 1$$

and for all i, j, m, n :

$$\sum_{k=1}^{d_i} \sum_{l=1}^{d_j} \alpha_0(\langle e_{ik} | e_{jl} \rangle_A) \cdot \left(1 \otimes \pi_0 \left(u_{nl}^{(j)*} x_{mk}^{(i)} \right) \right) = \langle e_{im} | e_{jn} \rangle_A \otimes 1,$$

where $x^{(i)} = (s^{(i)})^{-1} u^{(i)t} s^{(i)}$ is the inverse of $\overline{u^{(i)}}$. Thus (α_0, β_0) is a filtration preserving coaction of Q_0 on E .

It remains to see that Q_0 is in fact an initial object in the category $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. Let $I \subset \mathcal{U}$ be a Woronowicz C^* -ideal such that there exists a filtration preserving coaction (α_I, β_I) of \mathcal{U}/I on E such that β_I extends $(id \otimes \pi_I) \circ \beta_u$. We get in particular $I \in \mathcal{J}$, thus $I_0 \subset I$ and $p_I : Q_0 \rightarrow \mathcal{U}/I$ is then a morphism in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$.

Such a morphism is unique. Indeed, if η is a morphism from Q_0 to \mathcal{U}/I then $(id \otimes \eta) \circ \beta_0 = \beta_I$, so for all i, j, k , $\eta \circ \pi_0(u_{ij}^{(k)}) = \pi_I(u_{ij}^{(k)})$. Hence $\eta \circ \pi_0 = \pi_I$, and $\eta = p_I$ follows from uniqueness in the factorization theorem.

Finally, according to lemmas 3.4 and 3.6, we conclude that $(Q_0, \Delta_0, \alpha_0, \beta_0)$ is an initial object in $\mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$. \square

Remarks 3.8 – As in [4], we can make the following remarks:

- If $Q \in \mathcal{C}(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ coacts faithfully on E , then the morphism $\mu : Q_0 \rightarrow Q$ is onto. So that the quantum group associated with Q is a quantum subgroup of the one associated with Q_0 .
- If $(W_j)_{j \in \mathcal{J}}$ is a subfiltration of $(V_i)_{i \in \mathcal{I}}$ (that is $(W_j)_{j \in \mathcal{J}}$ is an orthogonal filtration of E , such that $\forall j \in \mathcal{J}$, there exists $i \in \mathcal{I}$ such that $W_j \subset V_i$) then the quantum symmetry group of $(E, \tau, (W_j)_{j \in \mathcal{J}}, J, \xi_0)$ is a quantum subgroup of the quantum symmetry group of $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$.

4. EXAMPLES

4.1 Example of a C^* -algebra equipped with an orthogonal filtration

We recall from [4] the construction of the quantum symmetry group of a C^* -algebra equipped with an orthogonal filtration.

Definition 4.1 – Let $(A, \tau, (V_i)_{i \in \mathcal{I}})$ be a C^* -algebra equipped with an orthogonal filtration (see Example 2.5 for the definition). We say that a Woronowicz C^* -algebra Q coacting on A coacts in a filtration preserving way, if the coaction $\alpha : A \rightarrow A \otimes Q$ of Q on A satisfies for all $i \in \mathcal{I}$, $\alpha(V_i) \subset V_i \odot Q$.

Theorem 4.2 ([4]) – *Let $(A, \tau, (V_i)_{i \in \mathcal{I}})$ be a C^* -algebra equipped with an orthogonal filtration. The category of Woronowicz C^* -algebras coacting on A in a filtration preserving way admits an initial object. The quantum group corresponding to that universal object is called the **quantum symmetry group** of $(A, \tau, (V_i)_{i \in \mathcal{I}})$.*

Setting $E = A$, $\xi_0 = 1_A$ and $J = a \mapsto a^*$, it is easy to see that the quantum symmetry group of $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, \xi_0)$ coincides with the one constructed in the previous theorem (if (Q, Δ, α) coacts on $(A, \tau, (V_i)_{i \in \mathcal{I}})$ in a filtration preserving way then $(\tau \otimes id) \circ \alpha = \tau(\cdot)1_A$ is automatic since $V_0 = \mathbb{C} \cdot 1_A$).

In fact our construction allows to see that the category of Woronowicz C^* -algebras coacting on $(A, \tau, (V_i)_{i \in \mathcal{I}})$ in a filtration preserving way admits an initial object, even when the assumption “ \mathcal{A}_0 is a $*$ -subalgebra of A ” is dropped.

In particular, we see that our construction generalizes the one of [6] in the sense that if $(\mathcal{A}, \mathcal{H}, D)$ is an admissible spectral triple and if we set:

- $E = A = \overline{\mathcal{A}}^{\mathcal{L}(\mathcal{H})}$,
- $\tau = \begin{cases} a \mapsto \frac{Tr_\omega(a|D|^{-p})}{Tr_\omega(|D|^{-p})} & \text{if } \mathcal{H} \text{ is infinite dimensional,} \\ \text{the usual trace otherwise,} \end{cases}$
where Tr_ω denotes the Dixmier trace and p is the metric dimension of $(\mathcal{A}, \mathcal{H}, D)$,
- the $(V_i)_{i \in \mathbb{N}}$ are the eigenspaces of the ‘noncommutative Laplacian’,
- ξ_0 and J are respectively the unit and the involution of A ,

then we recover the quantum isometry group of $(\mathcal{A}, \mathcal{H}, D)$ in the sense of [6].

Given a spectral triple, we have seen in example 2.5 another way to attach an Hilbert module equipped with an orthogonal filtration to it (induced by D instead of the Laplacian). For an admissible spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfying conditions of example 2.5.(4), the quantum symmetry group of \mathcal{H}^∞ and the quantum isometry group of $(\mathcal{A}, \mathcal{H}, D)$ in the sense of Goswami both exist. We do not know if they coincide in this situation. But in the case of the spectral triple of a Riemannian compact manifold the question is partially solved in the next paragraph.

4.2 Example of the bundle of exterior forms on a Riemannian manifold

Let M be a compact Riemannian manifold. Set $A = C(M)$, $\tau = \int \cdot d\text{vol}$ where $d\text{vol}$ denotes the Riemannian density of M and set $E = \Gamma(\Lambda^* M)$ equipped with its canonical Hilbert $C(M)$ -module structure. We denote by $D = \overline{d + d^*} : L^2(\Lambda^* M) \rightarrow L^2(\Lambda^* M)$ the de Rham operator. D is self-adjoint and has compact resolvent. So that $\text{sp}(D)$ can be written as: $\text{sp}(D) = \{\lambda_i ; i \in \mathbb{N}\}$, with $\lim_{i \rightarrow +\infty} |\lambda_i| = +\infty$ and where each λ_i is a real eigenvalue of D with finite multiplicity. For $i \in \mathbb{N}$ we denote by V_i the subspace associated with λ_i and by d_i the dimension of V_i . Note that $V_i \subset \Gamma^\infty(\Lambda^* M)$, so $V_i \subset E$.

Clearly, the family $(V_i)_{i \in \mathbb{N}}$ is an orthogonal filtration of E , E is full and $\mathcal{H} = L^2(\Lambda^* M)$.

We denote by $\xi_0 = m \mapsto 1_{\Lambda_m^* M} \in E$ and by $J : E \rightarrow E$ the canonical involution.

Comparison with the quantum isometry group of M as defined in [6]:

Let (α, β) be a filtration preserving coaction of a Woronowicz C^* -algebra Q on E . For ϕ a state on Q , the map $\text{id} \otimes \phi : L^2(\Lambda^* M) \odot Q \rightarrow L^2(\Lambda^* M)$ extends to a map $\text{id} \otimes \phi : L^2(\Lambda^* M) \otimes Q \rightarrow L^2(\Lambda^* M)$. We set $\beta_\phi = (\text{id} \otimes \phi) \circ \beta : L^2(\Lambda^* M) \rightarrow L^2(\Lambda^* M)$. Since β preserves the filtration, β_ϕ commutes with D on \mathcal{E}_0 . This implies that $\forall k \in \mathbb{N}$, $\beta_\phi(\text{Dom}(D^k)) \subset \text{Dom}(D^k)$ and $\beta_\phi \circ D^k = D^k \circ \beta_\phi$ on $\text{Dom}(D^k)$. Thus $\beta_\phi(\Gamma^\infty(\Lambda^* M)) \subset \Gamma^\infty(\Lambda^* M)$ and β_ϕ commutes with D^2 on $\Gamma^\infty(\Lambda^* M)$. Now for $f \in C^\infty(M)$, we have $\beta_\phi(f) = \beta_\phi(f \cdot \xi_0) = \alpha_\phi(f) \cdot \xi_0 = \alpha_\phi(f)$, where $\alpha_\phi = (\text{id} \otimes \phi) \circ \alpha$. Thus $\alpha_\phi(C^\infty(M)) \subset C^\infty(M)$ and α_ϕ commutes with \mathcal{L} (the Laplacian on functions) on $C^\infty(M)$. This shows that α is an isometric coaction of Q on $C(M)$ in the sense of [6]. In particular, the quantum symmetry group of $(\Gamma(\Lambda^* M), \tau, (V_i)_{i \in \mathbb{N}}, J, \xi_0)$ is a quantum subgroup of the quantum isometry group of M in the sense of [6]. We do not know if these quantum groups are in fact equal.

Comparison with the isometry group of M :

Let G be a compact group and let $\gamma : M \times G \rightarrow M$ be an isometric action of G on M . Then it is well known that

$$\begin{aligned} \alpha : C(M) &\rightarrow C(M \times G) \cong C(M) \otimes C(G) \\ f &\mapsto f \circ \gamma \end{aligned}$$

is a coaction of $C(G)$ on $C(M)$.

For $g \in G$ we set

$$\begin{aligned} \gamma_g : M &\rightarrow M & \text{and} & & \beta : \Gamma(\Lambda^* M) &\rightarrow C(G, \Gamma(\Lambda^* M)) \cong \Gamma(\Lambda^* M) \otimes C(G) \\ m &\mapsto mg & & & \omega &\mapsto (g \mapsto \gamma_g^*(\omega)) \end{aligned}$$

where $\gamma_g^* : \Gamma(\Lambda^* M) \rightarrow \Gamma(\Lambda^* M)$ denotes the pullback by γ_g . It is then easy to see that (α, β) is a filtration preserving coaction of $C(G)$ on $\Gamma(\Lambda^* M)$.

Assume conversely that (α, β) is a filtration preserving coaction of a *commutative* Woronowicz C^* -algebra Q on $\Gamma(\Lambda^* M)$. We will show that α and β are of the previous form, so that they arise from an action of a compact group G on M . Since α is a coaction of Q on $C(M)$, there exists a

compact group G such that $Q \cong C(G)$ and a continuous action $\gamma : M \times G \rightarrow M$ of G on M such that:

$$\begin{aligned}\alpha &= C(M) \rightarrow C(M \times G) \cong C(M) \otimes C(G) \\ f &\mapsto f \circ \gamma.\end{aligned}$$

For $g \in G$ we set $\beta_g = (id \otimes ev_g) \circ \beta$ and $\alpha_g = (id \otimes ev_g) \circ \alpha = f \mapsto f \circ \gamma_g$. We have seen in the previous paragraph that $C^\infty(M)$ is stable under α_g and that α_g commutes with the Laplacian on $C^\infty(M)$, so that γ_g is an isometry of M . Now we have to check that $\beta_g = \omega \mapsto \gamma_g^*(\omega)$. We already know that β_g and α_g coincide on $C(M)$, so $\beta_g = \omega \mapsto \gamma_g^*(\omega)$ on $C(M)$.

Then we show by induction on $k \in \mathbb{N}$ that $\beta_g(\omega) = \gamma_g^*(\omega)$ for all $\omega \in \Gamma(\Lambda^k M)$: Let $k \in \mathbb{N}$ such that for all $l \leq k$ and all $\omega \in \Gamma(\Lambda^l M)$, $\beta_g(\omega) = \gamma_g^*(\omega)$. Let f_0 be in $C(M)$ and f_1, \dots, f_{k+1} be in $C^\infty(M)$. Then:

$$\begin{aligned}\beta_g(df_1 \wedge \dots \wedge df_{k+1}) + \beta_g(d^*(f_1 df_2 \wedge \dots \wedge df_{k+1})) &= \beta_g(D(f_1 df_2 \wedge \dots \wedge df_{k+1})) \\ &= D(\beta_g(f_1 df_2 \wedge \dots \wedge df_{k+1})) = D\gamma_g^*(f_1 df_2 \wedge \dots \wedge df_{k+1}) \\ &= \gamma_g^* D(f_1 df_2 \wedge \dots \wedge df_{k+1}) \\ &= \gamma_g^*(df_1 \wedge \dots \wedge df_{k+1}) + \gamma_g^*(d^*(f_1 df_2 \wedge \dots \wedge df_{k+1})).\end{aligned}$$

But since $d^*(f_1 df_2 \wedge \dots \wedge df_{k+1})$ is a $(k-1)$ -form, we have:

$$\beta_g(d^*(f_1 df_2 \wedge \dots \wedge df_{k+1})) = \gamma_g^*(d^*(f_1 df_2 \wedge \dots \wedge df_{k+1}))$$

by the induction hypothesis. Thus $\beta_g(df_1 \wedge \dots \wedge df_{k+1}) = \gamma_g^*(df_1 \wedge \dots \wedge df_{k+1})$. Finally:

$$\beta_g(f_0 df_1 \wedge \dots \wedge df_{k+1}) = (f_0 \circ \gamma_g) \gamma_g^*(df_1 \wedge \dots \wedge df_{k+1}) = \gamma_g^*(f_0 df_1 \wedge \dots \wedge df_{k+1}).$$

But any $(k+1)$ -form is a linear combination of such forms, so that $\forall \omega \in \Gamma(\Lambda^{k+1} M)$, $\beta_g(\omega) = \gamma_g^*(\omega)$, which ends the induction.

Thus each isometric action of a compact group on M leads to a filtration preserving coaction of a commutative Woronowicz C^* -algebra on $\Gamma(\Lambda^* M)$ and conversely. This shows that the quantum symmetry group of $\Gamma(\Lambda^* M)$ might be a coherent quantum analog of the isometry group of M .

4.3 Basic example: free orthogonal quantum groups

Let n be in \mathbb{N} . We set $A = \mathbb{C}$, $E = \mathbb{C}^n$ equipped with its canonical Hilbert space structure, $\xi_0 = 0$ and $V_0 = \mathbb{C}^n$. Let $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be any invertible antilinear map. We denote by P the matrix of J in the canonical basis and by $\mathcal{A}_o(P)$ the universal Woronowicz C^* -algebra generated by a multiplicative and unitary matrix $u = (u_{ij})_{1 \leq i, j \leq n}$, satisfying the relation $u = P \bar{u} P^{-1}$ (the quantum group associated with $\mathcal{A}_o(P)$ is a so-called *free orthogonal quantum group*, see [2]). We denote by $\alpha_P : \mathbb{C} \rightarrow \mathbb{C} \otimes \mathcal{A}_o(P)$ the trivial coaction and by $\beta_P : \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathcal{A}_o(P)$ the linear map given by $\beta(e_i) = \sum_{k=1}^n e_k \otimes u_{ki}$ where $(e_k)_{1 \leq k \leq n}$ is the canonical basis of \mathbb{C}^n . We can easily check that $(id_{\mathbb{C}}, (V_0), J, \xi_0)$ is an orthogonal filtration of E and that (α_P, β_P) is a coaction of $\mathcal{A}_o(P)$ on

E . To see that (α_P, β_P) is a filtration preserving coaction, the only nontrivial point is to check that $(J \otimes *) \circ \beta_P = \beta_P \circ J$. We have for all i in $\{1, \dots, n\}$:

$$\begin{aligned} (J \otimes *) \circ \beta_P(e_i) &= \sum_{k=1}^n J(e_k) \otimes u_{ki}^* = \sum_{k,l=1}^n P_{lk} e_l \otimes u_{ki}^* = \sum_{l=1}^n e_l \otimes \left(\sum_{k=1}^n P_{lk} u_{ki}^* \right) \\ &= \sum_{l=1}^n e_l \otimes \left(\sum_{k=1}^n u_{lk} P_{ki} \right) \quad (\text{since } P\bar{u} = uP) \\ &= \sum_{k=1}^n P_{ki} \beta_P(e_k) = \beta_P \circ J(e_i). \end{aligned}$$

So (α_P, β_P) is a filtration preserving coaction of $\mathcal{A}_o(P)$ on \mathbb{C}^n . Now we show that it is a universal object in the category $\mathcal{C}(E, id_{\mathbb{C}}, (V_0), J, \xi_0)$. Let (α, β) be a filtration preserving coaction of a Woronowicz C^* -algebra Q on E and let $v = (v_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(Q)$ be characterized by $\beta(e_i) = \sum_{k=1}^n e_k \otimes v_{ki}$. By lemma 3.4 we already know that v is unitary. Furthermore, by a similar computation to the previous one, we see that $(J \otimes *) \circ \beta = \beta \circ J$ leads to the equality $P\bar{v} = vP$. Thus by universal property of $\mathcal{A}_o(P)$ we get the existence of a morphism $\mu : \mathcal{A}_o(P) \rightarrow Q$ such that for all $i, j \in \{1, \dots, n\}$, $\mu(u_{ij}) = v_{ij}$, which is clearly a morphism in the category $\mathcal{C}(E, id_{\mathbb{C}}, (V_0), J, \xi_0)$. Consequently the quantum symmetry group of E is the free orthogonal quantum group associated with P .

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REFERENCES

- [1] S. Baaj and G. Skandalis. C^* -algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory*, 2:683–721, 1989.
- [2] T. Banica. Théorie des représentations du groupe quantique compact libre $O(n)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(3):241–244, 1996.
- [3] T. Banica and J. Bichon. Quantum automorphism groups of vertex-transitive graphs of order ≤ 11 . *J. Algebraic Combin.*, 26(1):83–105, 2007.
- [4] T. Banica and A. Skalski. Quantum symmetry groups of C^* -algebras equipped with orthogonal filtrations. To appear in *Proceedings of the London Mathematical Society*, 2012.
- [5] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.
- [6] D. Goswami. Quantum group of isometries in classical and noncommutative geometry. *Comm. Math. Phys.*, 285(1):141–160, 2009.
- [7] E. C. Lance. *Hilbert C^* -modules, a toolkit for operator algebraists*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge university press, 1995.
- [8] B. R. Li. *Introduction to operator algebras*. World Scientific Publishing Co. Inc., River Edge, NJ, 1992.
- [9] A. Maes and A. Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk. (4)*, 16(1-2):73–112, 1998.

- [10] P. Podleś. Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups. *Comm. Math. Phys.*, 170(1):1–20, 1995.
- [11] A. Van Daele and S. Wang. Universal quantum groups. *Internat. J. Math.*, 7(2):255–263, 1996.
- [12] S. Wang. Free products of compact quantum groups. *Comm. Math. Phys.*, 167(3):671–692, 1995.
- [13] S. Wang. Quantum symmetry groups of finite spaces. *Comm. Math. Phys.*, 195(1):195–211, 1998.
- [14] N. E. Wegge-Olsen. *K-Theory and C^* -Algebras : A Friendly Approach*. Oxford University Press, USA, 1993.
- [15] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.
- [16] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.